Nets of conics containing at least one double line in $\mathrm{PG}(2, q)$, q even

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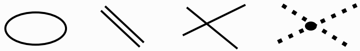
LINEAR SYSTEMS OF CONICS IN $\mathrm{PG}(2, q)$:

Non-empty conics in $\mathrm{PG}(2, q)$:



Linear systems of conics := **Subspaces**(PG(2-forms in the projective plane)):

- a pencil of conic $\mathcal{P} = \langle C_1, C_2 \rangle$.
- a net of conics $\mathcal{N} = \langle C_1, C_2, C_3 \rangle$.
- a web of conics $\mathcal{W} = \langle C_1, C_2, C_3, C_4 \rangle$.
- a squab of conics $\mathcal{W} = \langle C_1, C_2, C_3, C_4, C_5 \rangle$.

$$\text{Base points } := \bigcap_i C_i.$$

$\mathrm{PGL}(3, q)$-ORBITS OF LINEAR SYSTEMS OF CONICS

Purely algebraic approach:

- [Dickson, 1908]: pencils of conics over $\mathbb{F}_q$, q odd.
- [Wilson, 1914]: nets with at least a // over $\mathbb{F}_q$, q odd (Partial).
- [Campbell, 1927]: pencils of conics over $\mathbb{F}_q$, q even (Partial).
- [Campbell, 1928]: nets of conics over $\mathbb{F}_q$, q even (Partial).

Algebraic-geometric-combinatorial approach:

- [A., Lavrauw and Popiel, 2022]: pencils of conics over $\mathbb{F}_q$, q even.
- [Lavrauw, Popiel, Sheekey, 2020], [A. and Lavrauw, 2023]: non-empty base nets of conics for all q.
- [A. and Lavrauw, 2024] : webs and squabs of conics for all q.

Open problem

Classification of nets of conics in $\mathrm{PG}(2,q)$ with an empty base.

Connection:

- Tensors in $\mathbb{F}_q^3 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$.
- $\mathbb{F}_q$-linear symmetric rank-distance codes in $M_{3\times 3}(\mathbb{F}_q)$.

Main Theorem:

of $\mathrm{PGL}(3,q)$-orbits of nets with at least one $//$ is 18, q even.

LINEAR SYSTEMS AND SUBSPACES OF $\mathrm{PG}(5, q)$

- The Veronese surface $\mathcal{V}(\mathbb{F}_q)$ is the image of the Veronese embedding

$$\nu : (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$

- $C = \mathcal{Z}(a_{00}X_0^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2)$ $\iff H[a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{22}] \cap \mathcal{V}(\mathbb{F}_q)$.
- $K :=$ Subgroup of $\mathrm{PGL}(6, q)$ stabilising $\mathcal{V}(\mathbb{F}_q)$.
- $\mathrm{PGL}(3, q)$-orbits of linear systems $\iff$ K-orbits of subspaces of $\mathrm{PG}(5, q)$.

- pencils of conic in $\mathrm{PG}(2, q)$ $\iff$ solids in $\mathrm{PG}(5, q)$.
- nets of conics in $\mathrm{PG}(2, q)$ $\iff$ planes in $\mathrm{PG}(5, q)$.
- webs of conics in $\mathrm{PG}(2, q)$ $\iff$ lines in $\mathrm{PG}(5, q)$.
- squabs of conics in $\mathrm{PG}(2, q)$ $\iff$ points in $\mathrm{PG}(5, q)$.

Representation of Subspaces of $\mathrm{PG}(5,q)$

- $\mathrm{PG}(5,q) = \langle \mathcal{V}(\mathbb{F}_q) \rangle$.
- Every point $P = (x_0, .., x_5) \in \mathrm{PG}(5,q)$ can be represented by

$$M_P = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix}$$

- $rank(P) = rank(M_P)$.
- The plane in $\mathrm{PG}(5,q)$ spanned by the first three points of the standard frame is

$$\begin{bmatrix} x & y & z \\ y & \cdot & \cdot \\ z & \cdot & \cdot \end{bmatrix} := \{ \begin{bmatrix} x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} : (x,y,z,t) \in \mathbb{F}_q^4;\ (x,y,z) \neq (0,0,0) \}.$$

- The cubic curve associated with a plane π is

$$\mathrm{C} = \mathcal{Z}(\text{determinant of its matrix representation}).$$

ORBITS INVARIANTS:

Let W be a subspace of $\mathrm{PG}(5,q)$.
Let U_1, U_2,...,U_m denote the distinct K-orbits of r-spaces in $\mathrm{PG}(5,q)$.

The r-space orbit-distribution of W:

$$OD_r(W) := [u_1, u_2, \ldots, u_m],$$

where

u_i = # of r-spaces incident with W which belong to the orbit U_i.

POINTS AND HYPERPLANES OF $\mathrm{PG}(5, q)$

K-orbits of points; q even:

- $\mathcal{P}_1 :=$ Rank-one points.
- $\mathcal{P}_{2,n} :=$ Rank-two points in the *nucleus plane*.
- $\mathcal{P}_{2,s} :=$ Rank-two points outside the nucleus plane.
- $\mathcal{P}_3 :=$ Rank-three points.

K-orbits of Hyperplanes:

- $\mathcal{H}_1$:={ Hyperplanes $\iff$ repeated lines in $\mathrm{PG}(2, q)$}.
- $\mathcal{H}_{2,r}$:= { Hyperplanes $\iff$ pairs of real lines in $\mathrm{PG}(2, q)$}.
- $\mathcal{H}_{2,i}$:= { Hyperplanes $\iff$ pairs of conjugate imaginary lines in $\mathrm{PG}(2, q^2)$}.
- $\mathcal{H}_3$:= { Hyperplanes $\iff$ non-singular conics in $\mathrm{PG}(2, q)$}.

Nets of conics with at least one double line

even vs odd:

- **q odd:** $\exists$ a polarity: *the set of conic planes of* $\mathcal{V}(\mathbb{F}_q) \rightarrow$ *the set of tangent planes of* $\mathcal{V}(\mathbb{F}_q)$.
  - $\mathcal{N} = \langle C_1, C_2, C_3 \rangle; C_1 = // \longrightarrow$
    $\pi = H_1 \cap H_2 \cap H_3 \xrightarrow{\text{polarity}}$
    $\pi' = \langle P_1, P_2, P_3 \rangle; P_1 \in \mathcal{V}(\mathbb{F}_q) \longrightarrow$
    nets with at least one $//$ $\Longleftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially
    $\Longleftrightarrow$ nets with a non-empty base.
  - $\#\,\mathrm{PGL}(3,q)$-orbits of nets with at least one $//$ (*rank-1 nets*) $= 15$.
    [Lavrauw, Popiel, Sheekey, 2020]

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    nets with at least one // $\Longleftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially
    $\Longleftrightarrow$ nets with a non-empty base.
  - # $\mathrm{PGL}(3, q)$-orbits of nets with at least one // (*rank-1 nets*) $= 15$.
    [Lavrauw, Popiel, Sheekey, 2020]
- **q even:** No such polarity. Further, in [A. and Lavrauw, 2023]:
  - nets with at least one // $\not\Longleftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially
    $\Longleftrightarrow$ nets with a non-empty base.
  - # $\mathrm{PGL}(3, q)$-orbits of nets with a non-empty base $= 15$.

$r_{2,n}(\pi) = h_1(\pi)$

Theorem:

The number of hyperplanes in $\mathcal{H}_1$ containing a plane π in $\mathrm{PG}(5,q)$, $q \geq 4$ even, is equal to $|\pi \cap \pi_{\mathcal{N}}|$.

Proof.

- $\pi_{\mathcal{N}} \subset H \in \mathcal{H}_1$.
- If $r_{2,n}(\pi) = 0$, then $h_1(\pi) = 0$, otherwise $|\pi \cap \pi_{\mathcal{N}}| \geq 1$.
- If $r_{2,n}(\pi) = q+1$ with $\pi = \langle \ell, P \rangle$, $\ell \subset \pi_{\mathcal{N}}$, $P \notin \pi_{\mathcal{N}}$ of rank $1, 2, 3$, then by [A. and Lavrauw, 2024], P lies on $q+1$ $H \in \mathcal{H}_1$, and since $\ell \subset H$ for all such H, we get $h_1(\pi) = q+1$.

- For π with $\pi \cap \pi_{\mathcal{N}} = \{P\}$, we have $h_1(\pi) \leq 1$, since $\pi \subset H_1, H_2 \in \mathcal{H}_1 \implies \pi, \pi_{\mathcal{N}} \subset H_1 \cap H_2 \to$ ※.
  - Count flags (π, H) with $\pi \cap \pi_{\mathcal{N}} = \{P\}$, $H \in \mathcal{H}_1$.
  - $\alpha = \#$ of $\pi \subset H \in \mathcal{H}_1$; $|\pi \cap \pi_{\mathcal{N}}| = 1$.
    $\beta = \#$ of $\pi \subset \mathrm{PG}(5, q)$; $|\pi \cap \pi_{\mathcal{N}}| = 1$.

$$\alpha = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - (q+1)\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - 1\right) - 1,$$

$$\beta = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q - (q+1)\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q - 1\right) - 1.$$

  - $\pi_1, \ldots, \pi_\beta$ are the planes in $\mathrm{PG}(5, q)$ which meet $\pi_{\mathcal{N}}$ in a point:

$$\sum_{i=1}^{\beta} h_1(\pi_i) = (q^2 + q + 1)\alpha.$$

  - Since $h_1(\pi_i) \leq 1$, we conclude $h_1(\pi) = 1$ for all π with $\pi \cap \pi_{\mathcal{N}} = \{P\}$. □

Study structure

- $\pi \cap \mathcal{V}(\mathbb{F}_q) \neq \emptyset$ and $\pi \cap \pi_{\mathcal{N}} \neq \emptyset$: 9 orbits [A. and Lavrauw, 2023].
- $\pi \cap \mathcal{V}(\mathbb{F}_q) = \emptyset$ and $\pi \cap \pi_{\mathcal{N}} \neq \emptyset$: 9 orbits:

Theorem:

Let π be a plane in $\mathrm{PG}(5, q)$ such that $\pi \cap \mathcal{V}(\mathbb{F}_q) = \emptyset$ and $\pi \cap \pi_{\mathcal{N}} \neq \emptyset$. Then, one of the following cases holds:

1. $\pi = \pi_{\mathcal{N}}$,
2. $\pi \cap \pi_{\mathcal{N}} = \ell \in o_{12,1}$.
   This defines two K-orbits of planes with $OD_0(\pi) = [0, q+1, 0, q^2]$ or $OD_0(\pi) = [0, q+1, q, q^2 - q]$,
3. $\pi \cap \pi_{\mathcal{N}} = P$.
   This defines
   (a) one K-orbit of planes with $OD_0(\pi) = [0, 1, 0, q^2 + q]$,
   (b) one K-orbit with $OD_0(\pi) = [0, 1, 3q, q^2 - 2q]$,
   (c) two K-orbits with $OD_0(\pi) = [0, 1, q, q^2]$, and
   (d) two K-orbits with $OD_0(\pi) = [0, 1, 2q, q^2 - q]$.

$\pi \cap \mathcal{V}(\mathbb{F}_q) = \emptyset$ AND $\pi \cap \pi_{\mathcal{N}} = P$

Ideas of the proof:

- $r_{2,n}(\pi) = h_1(\pi) \to \exists\, H(\pi) \in \mathcal{H}_1;\, H(\pi) \supset \pi$.
- $\exists\, H(P) \in \mathcal{H}_1;\, H(P) \cap \mathcal{V}(\mathbb{F}_q) = C(P)$.
- **Theorem:** If $\pi \cap \mathcal{V}(\mathbb{F}_q) = \emptyset$ and $\pi \cap \pi_{\mathcal{N}} = P$ with $H(P) = H(\pi)$, then $\pi \in \Sigma_{18} \cup \Sigma_{19} \cup \Sigma_{20}$, with $OD_0(\Sigma_{18}) = [0, 1, 0, q^2 + q]$, $OD_0(\Sigma_{19}) = [0, 1, 3q, q^2 - 2q]$, and $OD_0(\Sigma_{20}) = [0, 1, q, q^2]$.
- **Theorem:** If $\pi \cap \mathcal{V}(\mathbb{F}_q) = \emptyset$ and $\pi \cap \pi_{\mathcal{N}} = P$, with $H(P) \neq H(\pi)$, then $\pi \in \Sigma_{21} \cup \Sigma_{22} \cup \Sigma_{23}$, with $OD_0(\Sigma_{21}) = [0, 1, 2q, q^2 - q]$, $OD_0(\Sigma_{22}) = [0, 1, q, q^2]$ and $OD_0(\Sigma_{23}) = [0, 1, 2q, q^2 - q]$.

Core aspects of the classification:

- Existence.
- Uniqueness.
- Combinatorial–geometric invariant–based identification.

π^K	Representatives	$OD_0(\pi)$	π^K	Representatives	$OD_0(\pi)$
Σ_1	$\begin{bmatrix} x & y & \cdot \\ y & z & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	$[q+1, 1, q^2-1, 0]$	$\Sigma_{\mathcal{N}}$	$\begin{bmatrix} \cdot & x & y \\ x & \cdot & z \\ y & z & \cdot \end{bmatrix}$	$[0, q^2+q+1, 0, 0]$
Σ_3	$\begin{bmatrix} x & \cdot & z \\ \cdot & y & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[2, 1, 2q-2, q^2-q]$	Σ_{16}	$\begin{bmatrix} \cdot & x & z \\ x & z & y \\ z & y & \cdot \end{bmatrix}$	$[0, q+1, 0, q^2]$
Σ_4	$\begin{bmatrix} x & \cdot & z \\ \cdot & y & z \\ z & z & \cdot \end{bmatrix}$	$[2, 1, 2q-2, q^2-q]$	Σ_{17}	$\begin{bmatrix} \cdot & x & y \\ x & z & \cdot \\ y & \cdot & z \end{bmatrix}$	$[0, q+1, q, q^2-q]$
Σ_7	$\begin{bmatrix} x & y & z \\ y & \cdot & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[1, q+1, q^2-1, 0]$	Σ_{18}	$\begin{bmatrix} x & y & z \\ y & cz & x+z \\ z & x+z & \cdot \end{bmatrix}$	$[0, 1, 0, q^2+q]$
Σ_8	$\begin{bmatrix} x & y & \cdot \\ y & \cdot & z \\ \cdot & z & \cdot \end{bmatrix}$	$[1, q+1, q-1, q^2-q]$	Σ_{19}	$\begin{bmatrix} x & y & \cdot \\ y & y+z & z \\ \cdot & z & x \end{bmatrix}$	$[0, 1, 3q, q^2-2q]$
Σ_9	$\begin{bmatrix} x & y & \cdot \\ y & z & z \\ \cdot & z & \cdot \end{bmatrix}$	$[1, 1, 2q-1, q^2-q]$	Σ_{20}	$\begin{bmatrix} x & y & bx \\ y & cx+y+z & z \\ bx & z & x \end{bmatrix}$	$[0, 1, q, q^2]$
Σ_{10}	$\begin{bmatrix} x & y & \cdot \\ y & z & \cdot \\ \cdot & \cdot & z \end{bmatrix}$	$[1, 1, 2q-1, q^2-q]$	Σ_{21}	$\begin{bmatrix} x & x+az & \cdot \\ x+az & z & y \\ \cdot & y & \cdot \end{bmatrix}$	$[0, 1, 2q, q^2-q]$
Σ_{11}	$\begin{bmatrix} x & y & \cdot \\ y & z & z \\ \cdot & z & x+z \end{bmatrix}$	$[1, 1, q-1, q^2]$	Σ_{22}	$\begin{bmatrix} x & x+z & z \\ x+z & z & y \\ z & y & \cdot \end{bmatrix}$	$[0, 1, q, q^2]$
Σ_{15}	$\begin{bmatrix} x & y & z \\ y & z & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[1, 1, q-1, q^2]$	Σ_{23}	$\begin{bmatrix} x & az & x \\ az & z & y \\ x & y & \cdot \end{bmatrix}$	$[0, 1, 2q, q^2-q]$

Table 1: K-orbits of planes in $\mathrm{PG}(5, q)$ intersecting $\pi_{\mathcal{N}}$ in at least one point and their point-orbit distributions for $q > 2$ even. The parameter c in Σ_{18} is a non-admissible element in $\mathbb{F}_q$ such that $\mathrm{Tr}(c^{-1}) = \mathrm{Tr}(1)$. The parameters in Σ_{20} satisfy $b \neq 1$ and $\mathrm{Tr}\left(c/(1+b^2)\right) = 1$. The parameter a in Σ_{21} and Σ_{23} satisfies $\mathrm{Tr}(a) = 1$.

Final Comments

- $\mathcal{C}_{20}$ is the union of a line and an imaginary pair of lines, $\mathcal{C}_{22}$ is irreducible, $\mathcal{C}_{21}$ is the union of a line and a double line, and $\mathcal{C}_{23}$ is the union of a non-singular conic with its tangent line.

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- Our work completes the partial classification of nets of conics with at least one double line in [Campbell, 1928].
- **Definition:** An $\mathbb{F}_q$-linear code of min. distance d is *complete* if not contained in a larger $\mathbb{F}_q$-linear code with the same d.
- **CSRD:** complete symmetric rank-distance.
- **Geometric Interpretation**: A k-dimensional CSRD code in $M_{3\times 3}(\mathbb{F}_q)$ of minimum distance $d \iff$ a $(k-1)$-dimensional subspace of $\mathrm{PG}(5, q)$ of minimum rank d that is maximal for this property.

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- **Definition:** An $\mathbb{F}_q$-linear code of min. distance d is *complete* if not contained in a larger $\mathbb{F}_q$-linear code with the same d.
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- Planes in $\Sigma_{\mathcal{N}} \cup \Sigma_{16} \cup \Sigma_{18}$ correspond to 3-dimensional CSRD in $M_{3\times3}(\mathbb{F}_q)$ with $d = 2$ and q even.
- Planes in $\mathrm{PG}(5,q)$, q even, are extendable to solids of minimum rank 2, except for those in $\Sigma_{\mathcal{N}} \cup \Sigma_{16} \cup \Sigma_{18}$.

Thank you for your attention!

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